Eigenvalues and Eigenvectors Differential Equations X. Du

- Suppose you have a square matrix *A*. Let *A* represent a linear transformation that is multiplied by a vector *V* (i.e. let *A* be a transformation matrix upon another vector *V*).
  - If *AV* is parallel to *V* (i.e. the transformed vector is parallel to the original vector), then *V* is an **eigenvector** of *A*.
    - By convention, $V \neq \vec{0}$.
  - Consequently, the constant factor by which the magnitude of the vector has changed is the **eigenvalue** associated with *V* and *A*.
  - i.e. $AV = \lambda V$, where *V* represents the eigenvector and λ represents the associated eigenvalue.
- **Characteristic polynomial** $= \det(A - \lambda I)$
  - Where does this come from?
    - $AV = \lambda V \Rightarrow AV - \lambda V = \vec{0} \Rightarrow (A - \lambda I)V = \vec{0}$, where *I* is the identity matrix.
    - $(A - \lambda I)V = \vec{0}$ iff $\det(A - \lambda I) = 0$.
      - $\det(A - \lambda I)$ is the **characteristic polynomial** of *A*.
  - For a two-by-two matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
    - $\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ab - bc = 0$
    - $\lambda^2 - (a + d)\lambda + ab - bc = 0$
      - **Trace** $T = a + d$
      - **Determinant** $D = ad - bc$
      - Characteristic equation can be written as $\lambda^2 - T\lambda + D = 0$.
- How to find eigenvalues and eigenvectors:
    - 1. Find the characteristic polynomial of *A* (i.e. find $\det(A - \lambda I)$).
  - 2. Solve $\det(A - \lambda I) = 0$ to obtain a set of eigenvalues.
  - 3. For each eigenvalue, find an associated eigenvector by substituting back into the equation $(A - \lambda I)V = \vec{0}$ and solving the system of equations.
    - The system of equations should be redundant (i.e. each individual equation in the system should be linearly dependent on all the others).
    - Note: Any eigenvector will do, as every eigenvalue associated with a specific eigenvector will just be multiples of each other (i.e. they will have the same direction). But for practical purposes, most people choose the most simplified eigenvector (i.e. choose $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ over $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$).
- Applications
  - Solving systems of differential equations
  - Transforming images (e.g. scaling, rotating, etc.)
  - Vibration analysis
  - Computational chemistry
    - Schrödinger equation
    - Molecular orbital theory